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A NOTE ON A CLASSICAL GENERATING FUNCTION FOR THE JACOBI POLYNOMIALS

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Dedicated to Acad. Bogoljub Stanković
on the occasion of his 80th birthday

Abstract

A more general form for a classical generating function for the Jacobi polynomials is given.

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1. If $z \neq \pm 1$, then we define $l(1;z) : \zeta = 1 + t(1-z)$ and $l(-1;z) : \zeta = -1 - t(1+z)$ for $0 \leq t < \infty$ as well as

$$\left(\frac{1-\zeta}{1-z}\right)^{\alpha} := \exp\left\{\alpha \log \frac{1-\zeta}{1-z}\right\}$$

for $\zeta \in S(1;z) := \mathbb{C} \setminus l(1;z)$, and

$$\left(\frac{1+\zeta}{1+z}\right)^{\beta} := \exp\left\{\beta \log \frac{1+\zeta}{1+z}\right\}$$

for $\zeta \in S(-1;z) := \mathbb{C} \setminus l(-1;z)$, provided α and β are arbitrary complex numbers.

It is clear that $S(1;x) = \mathbb{C} \setminus [1,\infty), S(-1;x) = \mathbb{C} \setminus (-\infty,-1]$ for $x \in (-1,1)$ and, moreover, that

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$$\left(\frac{1-\zeta}{1-x}\right)^{\alpha}=\frac{(1-\zeta)^{\alpha}}{(1-x)^{\alpha}}\,,\ \left(\frac{1+\zeta}{1+x}\right)^{\beta}=\frac{(1+\zeta)^{\alpha}}{(1+x)^{\beta}}$$

for $\zeta \in S(1;x)\cap S(-1;x)=\mathbb{C}\setminus\{(-\infty,-1)]\cup[1,\infty)\}$ and $x\in(-1,1)$.

Proposition 1. *Let γ be a rectifiable Jordan curve such that $\gamma\cup int\gamma\subset\mathbb{C}\setminus l(1;z)\cup l(-1;z)$, where $z\neq -1,1$, and $ind(\gamma;z)=1$. Then*

$$P_n^{(\alpha,\beta)}(z)=\frac{1}{2\pi i}\int_{\gamma}\left\{\frac{\zeta^2-1}{2(\zeta-z)}\right\}^n\left(\frac{1-\zeta}{1-z}\right)^{\alpha}\left(\frac{1+\zeta}{1+z}\right)^{\beta}\frac{d\zeta}{\zeta-z}\,. \tag{1.1}$$

The above integral representation is a corollary of the Rodrigues formula for the Jacobi polynomials as well as of the Cauchy integral formulas for the derivatives of a holomorphic function.

2. A well-known fact is that there exists unique complex-valued function h holomorphic in the region $H=\mathbb{C}\setminus[-1,1]$, and such that $h^2(z)=z^2-1$ for $z\in H$ and $h(x)>0$ when $x>1$. Usually, the value of this function at any point $z\in H$ is denoted by $\sqrt{z^2-1}$. The function ω, defined in H as $\omega(z)=z+h(z)$, is also holomorphic in H. Moreover, $\omega(z)\neq 0$ and $(\omega(z)+(\omega(z))^{-1})/2=z$ when $z\in H$, i.e. $\omega(z)$ is an inverse of the Zhukovski function $z=(\omega+\omega^{-1})/2$. As it is well-known, the last one is univalent in the domain $D=\{\omega:|\omega|>1\}$ and maps it onto H. Hence, the function ω maps H onto D. Since $\lim_{z\to\infty}\omega(z)=\infty,\omega$ is a meromorphic function in the region $\overline{\mathbb{C}}\setminus[-1,1]$ with a (simple) pole at the point of infinity.

If $z\in H$, then we define the function $p(z.w)$ in the disk, $U(0;|\omega(z)|^{-1})$ by the requirements $p^2(z,w)=1-2zw+w^2$ and $p(z,0)=1$. We denote this function by $\sqrt{1-2zw+w^2}$. Its existence follows from the fact that the disk $U(0;|\omega(z)|^{-1})$ is a simply connected region and $1-2zw+w^2\neq 0$ whenever w is in this disk, and $z\in H$. Indeed, the equality $1-2zw+w^2=0$ implies $w=\omega(z)$ or $w=(\omega(z))^{-1}$ which is impossible.

Let us note that the function $1+p(z,w)$ does not vanish in the disk $U(0;|\omega(z)|^{-1})$ when $z\in H$. Indeed, the equality $p(z_0,w_0)=0$ yields that $w_0(w_0-2z_0)=0$ which contradicts to $p(z_0,0)=p(z_0,2z_0)=1$. Hence,

$$\zeta(w)=\frac{2z-w}{1+p(z,w)} \tag{2.1}$$

is a holomorphic function in the disk $U(0;|\omega(z)|^{-1})$ for $z\in H$.

If $w\neq 0$, then from (2.1) it follows that $\zeta(w)=(1-p(z,w))w^{-1}$ and, hence, the equalities hold:

$$(1-w+p(z,w))(1-\zeta(w))=2(1-z),(1+w+p(z,w))(1+\zeta(w))=2(1+z).$$

A direct verification shows that these equalities are still valid for $w = 0$. Moreover, as their implication we obtain that $1 - w + p(z, w) \neq 0$ and $1 + w + p(z, w) \neq 0$ for $z \in H$ and $w \in U(0; |\omega(z)|^{-1})$.

We define the function $P^{(\alpha,\beta)}(z, w)$ for $z \in H$ and $w \in U(0; |\omega(z)|^{-1}) \setminus \{0\}$ by

$$P^{(\alpha,\beta)}(z, w) = \frac{2^{\alpha+\beta}}{p(z, w)(1 - w + p(z, w))^{\alpha}(1 + w + p(z, w))^{\beta}}$$

$$= \frac{2^{\alpha+\beta}}{\sqrt{1 - 2zw + w^2}(1 - w + p(z, w)(1 + w + p(z, w))^{\beta}},$$

and assume that $P^{(\alpha,\beta)}(z, 0) \equiv 1$.

Proposition 2. *For $z \in H$ and $w \in U(0; |\omega(z)|^{-1})$ it holds*

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(z)w^n = P^{(\alpha)}(z, w). \tag{2.2}$$

P r o o f. We note that $P^{(\alpha,\beta)}(z, w)$, as a function of w, is holomorphic in the disk $U(0; |\omega(z)|^{-1})$ and, hence, by Taylor's theorem it has a power series representation centered at the origin, i.e.

$$P^{(\alpha,\beta)}(z, w) = \sum_{n=0}^{\infty} a_n^{(\alpha,\beta)}(z)w^n. \tag{2.3}$$

If $0 < r < |\omega(z)|^{-1}$, then for the coefficient in the right-hand side of (2.3) we have

$$a_n^{(\alpha,\beta)}(z) = \frac{1}{2\pi i}\int_{C(0;r)} \frac{P^{(\alpha,\beta)}(z, w)}{w^{n+1}}\, dw, \quad n = 0, 1, 2, \dots,$$

where $C(0; r)$ is the positively oriented circle centered at the origin and having radius r.

From $p^2(z, w) = 1 - 2zw + w^2$ it follows that $p'_w(z, 0) = -z$, and using (2.1) we find that $\zeta'(0) = -1 + z^2 \neq 0$ for $z \in H$. Hence, there exists a neighbourhood $U(0; \delta)$ with $0 < \delta < |\omega(z)|^{-1}$, where the function $\zeta(w)$ is univalent. Since $\zeta(0) = 0$, it is clear that for arbitrary $r \in (0, \delta)$ the image of the circle $C(0; r)$ by the map $\zeta(w)$ is a positively oriented rectifiable Jordan curve $\gamma(z; r)$ such that $\mathrm{ind}(\gamma(z; r); z) = 1$. Moreover, r can be chosen such that $\gamma(z; r) \cup \mathrm{int}\gamma(z; r) \subset H \cap S(1; z) \cap S(-1; z)$.

Using the representation (1.1) with $\gamma = \gamma(z; r)$ and the equalities

$$\frac{\zeta^2(w) - 1}{2(\zeta(w) - z)} = \frac{1}{w}, \quad \frac{\zeta'(w)}{\zeta(w) - z} = \frac{1}{wp(z, w)}, \quad w \in U(0; |\omega(z)|^{-1}) \setminus \{0\},$$

and denoting the variable ζ by w, we obtain

$$P_n^{(\alpha,\beta)}(z) = \frac{1}{2\pi i}\int_{C(0;r)} \frac{P^{(\alpha,\beta)}(z,w)}{w^{n+1}}\,dw, \quad n = 0, 1, 2, \ldots.$$

Hence, $a_n^{(\alpha,\beta)}(z) = P_n^{(\alpha,\beta)}(z)$ for $n = 0, 1, 2, \ldots$. Then from (2.3) it follows that (2.2) holds in the disk $U(0; |\omega(z)|^{-1})$.

3. Let $g(z)$ be the unique complex-valued function which is holomorphic in the region $G = \mathbb{C}\setminus\{(-\infty,-1]\cup[1,\infty)\}$, and such that $g^2(z) = z^2 - 1$ and $g(0) = 1$. The function $\tau(z)$, defined as $\tau(z) = z + ig(z)$, is holomorphic and nowhere vanishing in G and, moreover, $(\tau(z) + (\tau(z))^{-1})/2 = z$ for $z \in G$. Hence, $\tau(z)$ is an inverse of the Zhukovski function $z = (\tau + \tau^{-1})/2$ and as such it is univalent in the half-plane $\Im\tau > 0$ and maps it onto the region G. In particular, the image of the point i is the origin. More precisely, the image of the half-plane $\Im z > 0$ is the region determined by the inequalities $|\tau| < 1$ and $\Im\tau > 0$, while the image of the interval $(-1, 1)$ is the arc of the unit circle located in the half-plane $\Im\tau > 0$. The image of the half-plane $\Im z < 0$ is the region determined by $|\tau| > 1$ and $\Im\tau > 0$.

The proof of Proposition 2 leads to the following assertion:

Proposition 3. *The equality* (2.2) *holds for arbitrary* $z \in G$ *and* $w \in U(0; \rho(z))$,*where* $\rho(z) = \min(|\tau(z)|, |\tau(z)|^{-1})$.

A particular case of (2.2) is the representation

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x) w^n = P^{(\alpha,\beta)}(x, w),$$

which holds for $-1 < x < 1$ and $|w| < 1$ [1, 10.8, (29)]. Indeed, in this case we have that $|\tau(x)| = |x + i\sqrt{1-x^2}| = 1$.

References

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